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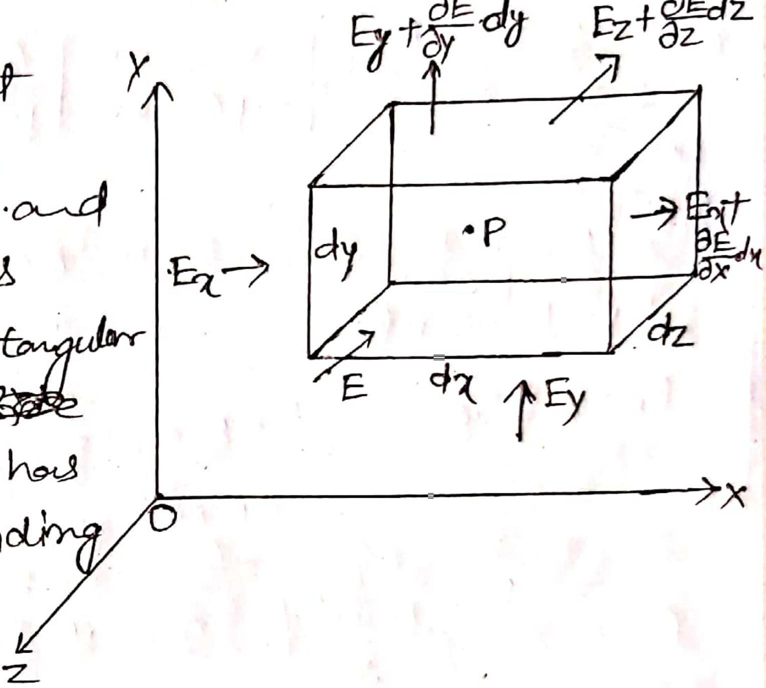
Poisson's and Laplace's equation for an electric field in Cartesian Co-ordinates.



Let us suppose that OX, OY and OZ are co-ordinates axes and P are P(x, y, z). Let us suppose that a rectangular parallelopiped of sides dx, dy and dz has been drawn surrounding the point P.

The face of this parallelopiped are parallel to the co-ordinate planes. Let us suppose that K be the dielectric constant of the medium in which it immersed and ρ, the volume density of charge near P.

Let D be the electric displacement at P. The components of D in the directions of OX, OY and OZ are D_x, D_y and D_z respectively and the corresponding electric intensity is D/K. Whose components in the above directions are E_x, E_y and E_z respectively. Thus $E_x = D_x/K$, $E_y = D_y/K$ and $E_z = D_z/K$. It has been assumed that K is dependent of direction. This means that the medium is homogeneous and isotropic. The front face is parallel to xy plane and its area is $dx \cdot dy$. The flux of the total normal electric displacement on it will be due to the z-components of D. If the point P lies at the centre of the parallelopiped, then the co-ordinates of the centre of the above face will be $\left(x, y, z + \frac{dz}{2}\right)$. The rate of change of D_z with respect to z will be $\left(D_z + \frac{\partial D_z}{\partial z} \cdot \frac{dz}{2}\right)$. Therefore, the flux of normal electric displacement on this face will be $\left(D_z + \frac{\partial D_z}{\partial z} \cdot \frac{dz}{2}\right) dx \cdot dy$.

Similarly the flux of normal electric displacement on the back face parallel to it will be $\left(D_z - \frac{\partial D_z}{\partial z} \cdot \frac{dz}{2}\right) dx \cdot dy$. Hence the net normal electric displacement in the direction parallel to OX,

$$\left(D_z + \frac{\partial D_z}{\partial z} \cdot \frac{dz}{2}\right) dx \cdot dy - \left(D_z - \frac{\partial D_z}{\partial z} \cdot \frac{dz}{2}\right) dx \cdot dy$$

$$= \frac{\partial D_z}{\partial z} \cdot dx \cdot dy \cdot dz$$

Similarly, the net flux of the total normal electric displacement on the faces perpendicular to OY and OZ axes will be $\frac{\partial D_x}{\partial x} \cdot dx \cdot dy \cdot dz$ and $\frac{\partial D_y}{\partial y} \cdot dx \cdot dy \cdot dz$ respectively. Hence the flux of total normal electric displacement on the whole surface of the parallelopiped will be

$$= \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) dx \cdot dy \cdot dz$$

∵ The volume of the parallelopiped $= dx \cdot dy \cdot dz$

∴ The total charge inside parallelopiped $= \rho\, dx \cdot dy\, dz$

But according to Gauss's theorem the total electric induction over the whole closed surface of the rectangular parallelopiped is equal to $\frac{1}{\epsilon_0}$ times the total charge enclosed by the surface

$$\therefore \left(\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z}\right) \cdot dx\, dy\, dz$$
$$= \rho\, dx \cdot dy \cdot dz$$

or, $$\frac{\partial D_x}{\partial x} + \frac{\partial D_y}{\partial y} + \frac{\partial D_z}{\partial z} = \rho/\epsilon_0 \quad \text{—— (1)}$$

But
$D_x = K E_x$, $D_y = K E_y$ and D_z $K E_z$

from equation (1)
$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho/\epsilon_0 K \quad \text{—— (2)}$$

In the vector form, equation (1) and (2) can be written as

$$\text{div}\,\vec{D} = \nabla \cdot \vec{D} = \rho/\epsilon_0 \quad \text{—— (3)}$$

$$\text{and div}\,\vec{E} = \nabla \cdot \vec{E} = \rho/\epsilon_0 K \quad \text{—— (4)}$$

for air $K = 1$

∴ equation (2) becomes
$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = \rho/\epsilon_0 \quad \text{—— (5)}$$

and equation (4) becomes

$$\text{div}\,\vec{E} = \nabla\cdot\vec{E} = \rho/\epsilon_0 \quad \text{———} (6)$$

If $v(x,y,z)$ be the potential at P, then

$$E_x = -\frac{\partial v}{\partial x}, \quad E_y = -\frac{\partial v}{\partial y} \text{ and } E_z = -\frac{\partial v}{\partial z}$$

∴ equation (2) becomes

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -\rho/\epsilon_0 K \quad \text{—} (7)$$

~~for~~ and equation (4) becomes

$$\text{div grad } v = \partial^2 v = -\rho\epsilon_0 k \quad \text{———} (8)$$

Similarly, equation (7) ad (8) reduce to

$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = -\rho/\epsilon_0 \quad \text{—} (9)$$

$$\text{and } \nabla^2 v = -\rho/\epsilon_0 \quad \text{———} (10)$$

for $K=1$ for air.

equation (2) (4), (7) ad (8) known as poission's equations expressed in different forms. This equation hold good for a homogenous isotropic dielectric medium. The expression in the L.H.S of equation (2) is called divergence E and hence from equation (4) Divergence is defined as the number of flux line originating from per unit volume. The notation ∇ is an operator and is pronounced as 'del' which means diffrentiation.

When $\rho=0$ i.e. when there is no charge in the field, equation (2), (4), (7) and (8) becoms

$$\frac{\partial E_x}{\partial x} + \frac{\partial E_y}{\partial y} + \frac{\partial E_z}{\partial z} = 0, \text{ or, } \nabla\cdot\vec{E} = 0 \quad \text{———} (11)$$

$$\text{and } \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} = 0, \text{ or. } \nabla^2 v = 0 \quad \text{——} (12)$$

There are laplacels equation.